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The dynamics of thin liquid jets in air

By V. M. ENTOV AND A. L. YARIN
Institute for Problems in Mechanics, Academy of Sciences of the USSR, Moscow

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The dynamics of propagation and disintegration of laminar liquid jets moving in air has been investigated theoretically. It is assumed that the jet is thin, i.e. the ratio of the characteristic transverse size to the longitudinal one is small. It is assumed also that the lateral surface of the jet is free of shearing forces and is 'almost free' of normal ones in the sense that the normal tractions other than isotropic pressure are small in comparison with the internal stresses acting in the jet cross-section.

Asymptotic quasi-one-dimensional equations of the continuity, momentum and moment of momentum of liquid motion in the jet have been derived. These equations were used as a basis for studying the process of growth of long-wave bending (transverse) disturbances of high-velocity jets of circular cross-section during their motion through air. The instability condition has been obtained and the growth rate of small bending disturbances of the jet has been found; the evolution of the jet shape at the stage of finite disturbances is investigated.

1. Introduction

The hydrodynamics of liquid jets has long attracted the attention of investigators. The phenomena of propagation and disintegration of jets arouse interest not only by their beauty but also by the possibility of wide application. The main objectives of the theory are to establish the character of disintegration of a jet and to calculate the length of its unbroken part and the size of drops into which the jet breaks up. These characteristics are determined by the mechanism of the growth of disturbances in the jet.

The capillary mechanism predominant in thin jets is associated with the intrinsic properties of a liquid: the disintegration is determined by the action of surface-tension forces, which tend to decrease the free surface by dividing the jet into drops. The results most important for describing this kind of disintegration were obtained by Rayleigh (1878) and Weber (1931).

Another mechanism is associated with the interaction of a jet with the ambient medium (e.g. air) and is predominant for high-velocity laminar jets which break up as a result of the growth of bending disturbances of the jet axis (Haenlein 1932; Tyler & Richardson 1925; Tyler & Watkin 1932; Grant & Middleman 1966; Sterling & Sleicher 1975).

The theoretical study of the dynamics of bending disturbances of liquid jets was initiated by Weber (1931), Middleman & Gavis (1965), Debye & Daen (1959) and Buckmaster (1973). This involves a very intricate problem of the dynamic effect of the air flow on the surface of a jet, unknown beforehand, the flow in which is also to be determined. The solution of the problem on the basis of general three-dimensional hydrodynamic equations involves great difficulties. Along these lines, therefore, it becomes possible to investigate only small deviations from the initial cylindrical shape of a jet, as was done by Debye & Daen (1959) for the inviscid liquid.

In the problems of the theory of liquid jets it is natural to use simplified quasi-one-dimensional equations which take into account the characteristic features of the flow, notably the jet slenderness. Such studies have appeared recently along with the theories of jet disintegration based on three-dimensional equations of fluid mechanics. Elements of the one-dimensional approach can be found in the papers by Trouton (1906), Weber (1931), Levi-Civita (1932), Ericksen (1952), and Levich (1962). One-dimensional equations were used by Kase & Matsuo (1965) and Matovich & Pearson (1969) to describe the flow in viscosity-dominated liquid filaments, by Lee (1974) to predict the capillary disintegration of ideal liquid jets and by Entov *et al.* (1980*a*, *b*) to investigate the disintegration of capillary jets of highly viscous and non-Newtonian liquids. In the papers by Green & Laws (1966, 1968), Green, Laws & Naghdi (1968), Green, Naghdi & Wenners (1974*a*, *b*), Green (1975, 1976) and Naghdi (1979) a theory has been developed which regards a liquid jet as a Cosserat line. In these papers a closed system of one-dimensional equations has been obtained for straight viscous liquid jets and for ideal liquid jets. This system was used by Bogy (1978*a*,*b*, 1979*a*,*b*) to describe the formation of satellite drops in the capillary disintegration of jets. There are also studies where one-dimensional equations are derived from the variational d'Alembert–Lagrange principle (Khusid 1979) and studies in which the jet slenderness is used to describe the flow in the boundary-layer approximation (Markova & Shkadov 1972).

The question of construction of one-dimensional equations suitable for the description of both capillary and bending disturbances remains open. The derivation of such a system of equations and its subsequent solution in the case of bending disturbances is the main objective of the present paper which is based on our previous works (Entov & Yarin, 1979, 1980).

2. Geometric and kinematic relations

Consider a smooth time-dependent three-dimensional curve $\Gamma(t)$ given parametrically by the equation $\boldsymbol{r} = \boldsymbol{R}(s,t), s_- \leqslant s \leqslant s_+$, where s is an arbitrary parameter and t is time, and call it the jet axis. With each point $O(s,t)$ of the curve we associate the normal plane yOz by directing the axis Oy along the unit principal-normal vector $\boldsymbol{n}$ and Oz along the unit binormal vector $\boldsymbol{b}$ to $\Gamma(t)$ at the point O. Consider in the plane of variables (y,z) a set of simply connected domains $\mathscr{D}(s,t)$ which is continuously dependent on s and t. The domain $\mathscr{D}(s,t)$ has the point O as the centre of gravity. Let us call $\mathscr{D}(s,t)$ the cross-section of the jet at a point s at an instant t. This definition implies that the cross-section of the liquid volume of the jet by a plane normal to the jet axis coincides with $\mathscr{D}(s,t)$ at the point s at the instant t. These conditions are sketched in figure 1.

If d is the diameter of the domain $\mathscr{D}$ then the jet is considered thin subject to the condition that

$$\epsilon = \max\left(\frac{d}{l}, kd, \kappa d\right) \ll 1, \tag{2.1}$$

where l is the lengthscale along the jet axis, k is the curvature and κ is the torsion of the axis. The position of a point (or a liquid particle) in the jet is determined by three parameters q^i, $i = 1, 2, 3$, $q^1 = y$, $q^2 = z$, $q^3 = s$, which serve as coordinates in

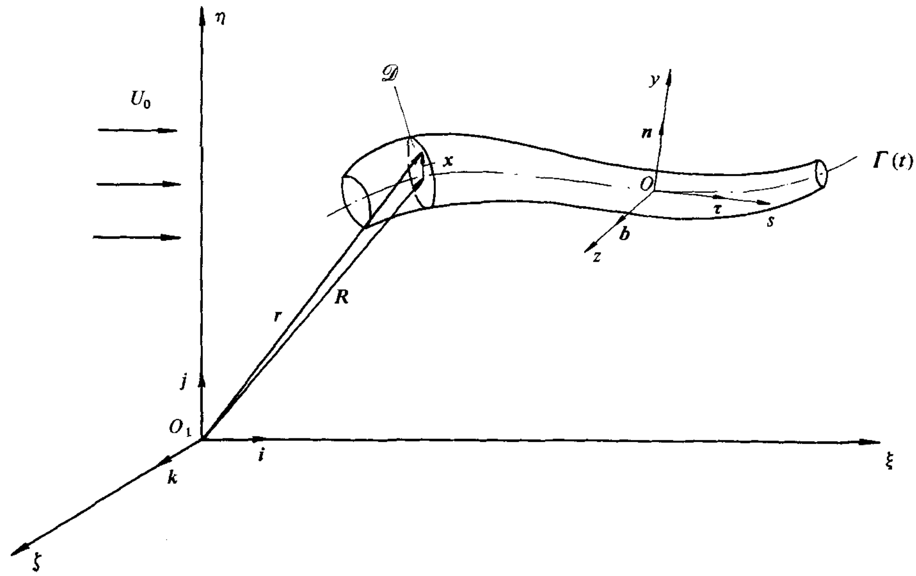

FIGURE 1. Sketch of the segment of the jet, with frame of reference.

a mobile curvilinear (non-orthogonal in the case $\kappa \neq 0$) coordinate system with contravariant basis vectors $\boldsymbol{a}^j$:

$$\left.\begin{aligned} \boldsymbol{r}(y,z,s,t) &= \boldsymbol{R}(s,t)+y\boldsymbol{n}(s,t)+z\boldsymbol{b}(s,t) \equiv \boldsymbol{R}(s,t)+\boldsymbol{x}, \\ \boldsymbol{a}^1 &= \boldsymbol{n}+\kappa z(1-ky)^{-1}\boldsymbol{\tau}, \quad \boldsymbol{a}^2 = \boldsymbol{b}-\kappa y(1-ky)^{-1}\boldsymbol{\tau}, \\ \boldsymbol{a}^3 &= \boldsymbol{\tau}\lambda^{-1}(1-ky)^{-1} \quad \lambda = \left|\frac{\partial \boldsymbol{R}}{\partial s}\right|. \end{aligned}\right\} \tag{2.2}$$

Here $\boldsymbol{\tau}$ is the unit tangent vector of the jet axis.

We use the following definition of the gradient tensor $\nabla \boldsymbol{w}$: $\mathrm{d}\boldsymbol{w} = (\nabla \boldsymbol{w})^{\mathrm{T}}\cdot \mathrm{d}\boldsymbol{x}$, where $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{x})$ is an arbitrary vector (another definition of the gradient tensor is also possible: $\mathrm{d}\boldsymbol{w} = \nabla \boldsymbol{w}\cdot \mathrm{d}\boldsymbol{x}$; see e.g. Astarita & Marrucci 1974, p. 21). This definition is convenient in that it enables one to introduce the dyadic gradient operator ∇

$$\nabla = \boldsymbol{a}^i\frac{\partial}{\partial q^i} = [\boldsymbol{n}+\kappa z(1-ky)^{-1}\boldsymbol{\tau}]\frac{\partial}{\partial y}+[\boldsymbol{b}-\kappa y(1-ky)^{-1}\boldsymbol{\tau}]\frac{\partial}{\partial z}+\lambda^{-1}(1-ky)^{-1}\boldsymbol{\tau}\frac{\partial}{\partial s}. \tag{2.3}$$

At each point of the liquid jet two velocities are defined: the velocity $\boldsymbol{u}$ of motion of a point with fixed coordinates (y,z,s) (the velocity of frame of reference associated with the jet) and the velocity $\boldsymbol{v}$ of motion of a liquid particle:

$$\left.\begin{aligned} \boldsymbol{u} &= \frac{\partial \boldsymbol{r}(q^i,t)}{\partial t}, \quad \boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}(q^i,t)}{\mathrm{d}t}, \\ \boldsymbol{u}(\boldsymbol{x},s,t) &= \boldsymbol{U}(s,t)+\boldsymbol{G}_u^*\cdot\boldsymbol{x} = \boldsymbol{U}(s,t)+y\frac{\partial \boldsymbol{n}}{\partial t}+z\frac{\partial \boldsymbol{b}}{\partial t}, \\ \boldsymbol{v}(\boldsymbol{x},s,t) &= \boldsymbol{V}(s,t)+\boldsymbol{G}_v^*\cdot\boldsymbol{x}+\boldsymbol{v}_2, \\ \boldsymbol{G}_u^* &= [\nabla \boldsymbol{u}]^{\mathrm{T}}_{y=z=0}, \quad \boldsymbol{G}_v^* = [\nabla \boldsymbol{v}]^{\mathrm{T}}_{y=z=0}, \\ \boldsymbol{U} &= \boldsymbol{u}(0,0,s,t) = \frac{\partial \boldsymbol{R}}{\partial t}, \quad \boldsymbol{V} = \boldsymbol{v}(0,0,s,t) = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t}. \end{aligned}\right\} \tag{2.4}$$

It can readily be seen that V is the velocity of the liquid in the centre of gravity of the jet cross-section and v_2 is the nonlinear part of the expansion of the velocity v in y and z. For slender jets and sufficiently viscous liquids the contribution of v_2 must be in a sense small for obvious reasons. So the velocity profile is represented to a high degree of accuracy as a superposition of translational motion together with the centre of gravity, rigid-body rotation around it and affine deformation.

The position of the jet axis can be described by various methods. In particular, let us assume that the jet motion is such that the tangent to the jet axis at any instant and at all points makes an acute angle with a straight line $O_1\xi$. Then we can introduce a Cartesian coordinate system $O_1\xi\eta\zeta$ with unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and describe the jet axis by the equations

$$\left.\begin{aligned} &\xi = s, \quad \eta = H(s,t), \quad \zeta = Z(s,t), \quad \boldsymbol{R} = \boldsymbol{i}s + \boldsymbol{j}H + \boldsymbol{k}Z, \\ &\boldsymbol{\tau} = \lambda^{-1}(\boldsymbol{i} + H_{,s}\boldsymbol{j} + Z_{,s}\boldsymbol{k}), \quad \lambda = (1 + H_{,s}^2 + Z_{,s}^2)^{\frac{1}{2}}, \end{aligned}\right\} \tag{2.5}$$

where H and Z are displacements of the jet axis in the directions $O_1\eta$ and $O_1\zeta$.

Next we have

$$\left.\begin{aligned} &\boldsymbol{R}_{,t} = \boldsymbol{U} = H_{,t}\boldsymbol{j} + Z_{,t}\boldsymbol{k}, \quad \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \boldsymbol{V} = \boldsymbol{U} + W\boldsymbol{\tau}, \\ &W = \lambda\frac{\mathrm{d}s}{\mathrm{d}t}, \quad \frac{\mathrm{d}s}{\mathrm{d}t} = \boldsymbol{V}\cdot\boldsymbol{i} \end{aligned}\right\} \tag{2.6}$$

(it can be shown that $\lambda\boldsymbol{V}\cdot\boldsymbol{i} = V_\tau - U_\tau$).

Hereinafter the projections of the vectors onto the directions of the unit vectors $\boldsymbol{n}$, $\boldsymbol{b}$ and $\boldsymbol{\tau}$ are marked by the corresponding subscripts.

Expressions (2.6) relate the evolution of the jet axis at any instant to the velocity field on the jet axis

$$\frac{\partial \boldsymbol{R}}{\partial t} = \boldsymbol{V} - \boldsymbol{\tau}\lambda(\boldsymbol{V}\cdot\boldsymbol{i}). \tag{2.7}$$

The relations (2.5) and the fourth equation in (2.6) explicitly use the parametrization of the jet axis chosen here. In certain circumstances a different choice of the parameter s is required (see §7), which leads to a change in the above-mentioned kinematic relations.

The derivatives $\partial\boldsymbol{n}/\partial t$, $\partial\boldsymbol{b}/\partial t$ and $\partial\boldsymbol{\tau}/\partial t$ can be related to the evolution of the radius vector of the jet axis $\boldsymbol{R}$ with the aid of simple geometric considerations and, consequently, to the velocity field on the jet axis by using the kinematic relation (2.7).

For the curvature and torsion of the jet axis there are relations known from differential geometry.

3. Dynamic equations

Now turn to the derivation of asymptotic equations of the continuity, momentum and moment of momentum of a liquid in the jet. The mass of liquid between two cross-sections s_1 and s_2 is

$$M = \int_{s_1}^{s_2}\left[\int_{\mathscr{D}} \rho g \,\mathrm{d}S\right]\mathrm{d}s, \quad g = \boldsymbol{a}_1\cdot[\boldsymbol{a}_2\times\boldsymbol{a}_3] = \lambda(1-ky), \tag{3.1}$$

where ρ is density. As a result of the motion of the jet cross-section with a fixed value of s, the transfer of mass (as well as momentum and moment of momentum) through it occurs at a velocity $(\boldsymbol{v}-\boldsymbol{u})\cdot\boldsymbol{\tau} = v_\tau - u_\tau$. The mass flux through the jet cross-section $\mathscr{D}(s,t)$ is

$$Q = \int_{\mathscr{D}} \rho(v_\tau - u_\tau)\,\mathrm{d}S. \tag{3.2}$$

The rate of change in the liquid mass enclosed between the cross-sections s_1 and s_2 is equal to the difference of the mass fluxes through these cross-sections. As s_2 tends to s_1, we obtain the differential continuity equation for the jet (subsequently the liquid is considered incompressible)

$$\frac{\partial \rho\lambda f}{\partial t} + \frac{\partial}{\partial s}\left[\rho \int_{\mathscr{D}} (v_\tau - u_\tau)\,\mathrm{d}S\right] = 0. \tag{3.3}$$

Here f is the area of the jet cross-section.

Now derive the momentum balance equation for the chosen element of the jet. For the momentum between the cross-sections s_1 and s_2 and the flux of momentum through the jet cross-section corresponding to the fixed value of s we have

$$\boldsymbol{J} = \int_{s_1}^{s_2}\left[\int_{\mathscr{D}} \rho \boldsymbol{v} g\,\mathrm{d}S\right]\mathrm{d}s, \quad \boldsymbol{L} = \int_{\mathscr{D}} \rho \boldsymbol{v}(v_\tau - u_\tau)\,\mathrm{d}S. \tag{3.4}$$

The stresses acting in the cross-section will be denoted by $\boldsymbol{\sigma}_\tau(\boldsymbol{x}, s, t)$. We also assume that external forces $\boldsymbol{F}$ per unit mass and external loads distributed over the lateral surface act on the liquid. The resultant action of external loads will be specified by the linear density of forces $\boldsymbol{q}$ applied to the jet axis and by moment $\boldsymbol{m}$ per unit length of the jet axis. Then the momentum equation takes the form

$$\frac{\partial}{\partial t}\left[\rho \int_{\mathscr{D}} \boldsymbol{v} g\,\mathrm{d}S\right] + \frac{\partial}{\partial s}\left[\rho \int_{\mathscr{D}} \boldsymbol{v}(v_\tau - u_\tau)\,\mathrm{d}S\right] = \frac{\partial}{\partial s}\left[\int_{\mathscr{D}} \boldsymbol{\sigma}_\tau\,\mathrm{d}S\right] + \rho f \lambda \boldsymbol{F} + \lambda \boldsymbol{q}. \tag{3.5}$$

The moment-of-momentum balance equation is derived in a similar manner:

$$\frac{\partial}{\partial t}\left[\int_{\mathscr{D}} (\boldsymbol{r}\times\rho\boldsymbol{v})\, g\,\mathrm{d}S\right] + \frac{\partial}{\partial s}\left[\int_{\mathscr{D}} (\boldsymbol{r}\times\rho\boldsymbol{v}(v_\tau - u_\tau))\,\mathrm{d}S\right]$$
$$= \frac{\partial}{\partial s}\left[\int_{\mathscr{D}} (\boldsymbol{r}\times\boldsymbol{\sigma}_\tau)\,\mathrm{d}S\right] + \lambda\boldsymbol{R}\times\boldsymbol{q} + \rho\lambda f\boldsymbol{R}\times\boldsymbol{F} + \lambda\boldsymbol{m} - \left[\rho\lambda k\int_{\mathscr{D}} y\boldsymbol{x}\,\mathrm{d}S\right]\times\boldsymbol{F}. \tag{3.6}$$

In calculating the moment of mass forces the condition

$$\int_{\mathscr{D}} \boldsymbol{x}\,\mathrm{d}S = \boldsymbol{0}$$

was used.

Let us form the vector product of $\boldsymbol{R}(s,t)$ with (3.5) and subtract it from (3.6). We thus obtain

$$\frac{\partial}{\partial t}\left[\lambda\rho\int_{\mathscr{D}} (\boldsymbol{x}\times\boldsymbol{v}(1-ky))\,\mathrm{d}S\right] + \boldsymbol{U}\times\rho\lambda\int_{\mathscr{D}} \boldsymbol{v}(1-ky)\,\mathrm{d}S$$
$$+\lambda\rho\boldsymbol{\tau}\times\int_{\mathscr{D}} \boldsymbol{v}(v_\tau - u_\tau)\,\mathrm{d}S + \rho\frac{\partial}{\partial s}\left[\int_{\mathscr{D}} (\boldsymbol{x}\times\boldsymbol{v}(v_\tau - u_\tau))\,\mathrm{d}S\right]$$
$$= \frac{\partial}{\partial s}\left[\int_{\mathscr{D}} (\boldsymbol{x}\times\boldsymbol{\sigma}_\tau)\,\mathrm{d}S\right] + \lambda\boldsymbol{\tau}\times\int_{\mathscr{D}} \boldsymbol{\sigma}_\tau\,\mathrm{d}S - \rho\lambda k\left[\int_{\mathscr{D}} y\boldsymbol{x}\,\mathrm{d}S\right]\times\boldsymbol{F} + \lambda\boldsymbol{m}. \tag{3.7}$$

The next step is to write (3.3), (3.5) and (3.7) asymptotically, retaining in them the dominant terms only.

Let us take instead of (2.4) a simplified representation of the distribution of liquid velocities. Namely, we shall assume that at each instant the instantaneous motion of the liquid cross-section, which coincides with the normal cross-section of the jet, reduces mainly to a combination of the translational motion with the centre of gravity, rigid-body rotation around it and isotropic expansion or contraction in the cross-sectional plane. This assumption about the character of deformation motion is an analogue of the hypothesis of flat cross-sections in the theory of bar bending. It is based physically on the fact that the jet is thin and its lateral surface is free from shearing forces.

This assumption may be violated in three main cases. The first one is represented by a thin liquid jet with elliptical cross-section. The capillary forces will obviously tend to transform the ellipse into a circle. The second case is the circular jet with convergent–divergent initial distribution of transverse velocities. In this case the circular cross-section will be transformed into the elliptical one. These two cases are typical for inviscid jets issuing from non-circular orifices. The interplay between capillary and inertial forces gives rise to oscillations, which will eventually decrease owing to viscous damping. These two cases are out of the scope of the present theory.

If the liquid viscosity and jet velocity are high enough so that the action of capillary forces is negligible, the present theory applies as well to jets of arbitrary cross-section with velocities initially aligned along the jet axis. In all cases only long-wave disturbances are considered.

The third case is more subtle. It corresponds to a jet compressed laterally along a diameter by opposite external loads tending to flatten it. But this flattening is counteracted by capillary forces for the thin jet, and in any case is much slower for viscous jets than jet bending and (or) uniform extension–compression of jet elements as soon as we consider long-wave modes of jet evolution. This is shown by direct calculation for Newtonian viscous liquids (see §8), but can be conjectured to be valid for sufficiently 'viscous' non-Newtonian liquids as well.

Taking into account the assumption made, we have instead of (2.4)

$$\left.\begin{aligned} \boldsymbol{v} &= \boldsymbol{V}+\boldsymbol{\Omega}\times\boldsymbol{x}+\delta\boldsymbol{x}+\boldsymbol{v}_2, \\ \boldsymbol{v}_2 &= \phi_1(y,z,s)\,\boldsymbol{n}+\phi_2(y,z,s)\,\boldsymbol{b}+\phi_3(y,z,s)\,\boldsymbol{\tau}, \\ \boldsymbol{u} &= \boldsymbol{U}+y\frac{\partial \boldsymbol{n}}{\partial t}+z\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{U}+\boldsymbol{\omega}\times\boldsymbol{x}. \end{aligned}\right\} \tag{3.8}$$

Here $\boldsymbol{\Omega}$ is the angular velocity of the liquid cross-section, and the functions ϕ_i may be expanded in series in y and z, starting from the second-order terms; the components of the vector $\boldsymbol{\omega}$ are determined using the expressions for $\partial\boldsymbol{n}/\partial t$ and $\partial\boldsymbol{b}/\partial t$.

The expansions (3.8) make it possible to evaluate the integrals occurring in the obtained balance equations of the jet dynamics. More specifically, for the mass flux through the jet cross-section we have

$$\int_{\mathscr{D}}(v_\tau-u_\tau)\,\mathrm{d}S = (V_\tau-U_\tau)f+\boldsymbol{\tau}\cdot\boldsymbol{j}_1, \quad \boldsymbol{j}_1 = \int_{\mathscr{D}}\boldsymbol{v}_2\,\mathrm{d}S, \quad |\boldsymbol{j}_1| = O(\epsilon^2 f|\boldsymbol{V}|). \tag{3.9}$$

The continuity equation (3.3), after neglecting the small term $j_{1\tau,s}$ and noting that $W = V_\tau - U_\tau$, is transformed into

$$\frac{\partial\lambda f}{\partial t}+\frac{\partial fW}{\partial s} = 0. \tag{3.10}$$

The balance equations of momentum and moment of momentum are transformed in a similar manner, and, after omitting the higher orders of small quantities, take the forms

$$\frac{\partial \lambda f V}{\partial t}+\frac{\partial f V W}{\partial s}=\frac{1}{\rho}\frac{\partial}{\partial s}(P\boldsymbol{\tau}+\boldsymbol{Q})+\lambda f \boldsymbol{F}+\frac{1}{\rho}\lambda \boldsymbol{q}, \tag{3.11}$$

$$\frac{\partial \lambda \boldsymbol{K}}{\partial t}+\lambda[\boldsymbol{U}\times \boldsymbol{j}_1+\boldsymbol{\tau}\times(U\boldsymbol{j}_{1\tau}+\boldsymbol{j}_2+W\boldsymbol{j}_1)-k\boldsymbol{U}\times(\boldsymbol{\Omega}\times\boldsymbol{j}_3+\delta\boldsymbol{j}_3)]$$
$$+\frac{\partial}{\partial s}(W\boldsymbol{K}_1+\boldsymbol{j}_4\times \boldsymbol{V})=\frac{1}{\rho}\frac{\partial \boldsymbol{M}}{\partial s}+\frac{1}{\rho}\lambda\boldsymbol{\tau}\times\boldsymbol{Q}-\lambda k\boldsymbol{j}_3\times\boldsymbol{F}+\frac{\lambda}{\rho}\boldsymbol{m}, \tag{3.12}$$

$$P=\boldsymbol{\tau}\cdot\int_{\mathscr{D}}\boldsymbol{\sigma}_\tau\,\mathrm{d}S,\quad \boldsymbol{Q}=\int_{\mathscr{D}}\boldsymbol{\sigma}_\tau\,\mathrm{d}S-P\boldsymbol{\tau},\quad \boldsymbol{M}=\int_{\mathscr{D}}\boldsymbol{x}\times\boldsymbol{\sigma}_\tau\,\mathrm{d}S, \tag{3.13}$$

$$\boldsymbol{K}=\int_{\mathscr{D}}(\boldsymbol{x}\times\boldsymbol{\Omega}\times\boldsymbol{x})\,\mathrm{d}S-k\boldsymbol{j}_3\times\boldsymbol{V}=\boldsymbol{n}(\Omega_n I_n-\Omega_b I_{bn}-kV_\tau I_{bn})$$
$$+\boldsymbol{b}(\Omega_b I_b-\Omega_n I_{bn}+kV_\tau I_b)+\boldsymbol{\tau}[\Omega_\tau(I_n+I_b)-kV_b I_b+kV_n I_{bn}], \tag{3.14}$$

$$\boldsymbol{K}_1=\int_{\mathscr{D}}(\boldsymbol{x}\times\boldsymbol{\Omega}\times\boldsymbol{x})\,\mathrm{d}S=\boldsymbol{n}(\Omega_n I_n-\Omega_b I_{bn})+\boldsymbol{b}(\Omega_b I_b-\Omega_n I_{bn})+\boldsymbol{\tau}\Omega_\tau(I_n+I_b), \tag{3.15}$$

$$\left.\begin{aligned} &I_n=\int_{\mathscr{D}}z^2\,\mathrm{d}S,\quad I_b=\int_{\mathscr{D}}y^2\,\mathrm{d}S,\quad I_{bn}=\int_{\mathscr{D}}zy\,\mathrm{d}S,\\ &\boldsymbol{j}_2=\int_{\mathscr{D}}(\boldsymbol{\Omega}\times\boldsymbol{x}+\delta\boldsymbol{x})\,[(\boldsymbol{\Omega}-\boldsymbol{\omega})\cdot(\boldsymbol{x}\times\boldsymbol{\tau})]\,\mathrm{d}S,\\ &\boldsymbol{j}_3=\int_{\mathscr{D}}\boldsymbol{x}y\,\mathrm{d}S=\boldsymbol{n}I_b+\boldsymbol{b}I_{bn},\quad \boldsymbol{j}_4=\int_{\mathscr{D}}\boldsymbol{x}[(\boldsymbol{\Omega}-\boldsymbol{\omega})\cdot(\boldsymbol{x}\times\boldsymbol{\tau})]\,\mathrm{d}S.\end{aligned}\right\} \tag{3.16}$$

The quantities $\boldsymbol{\tau}P$, $\boldsymbol{Q}$ and $\boldsymbol{M}$ represent respectively the longitudinal force, the shearing force and the moment of stresses in the jet cross-section.

4. Closure conditions

The closure of the obtained system of equations (3.10)–(3.12) involves the use of kinematic and geometric relations which relate the evolution of the jet-axis configuration to the velocity of liquid particles on the axis (see §2) and also the establishment of a connection between the velocity and stress fields in the jet with the liquid rheology taken into account.

We consider here only the case of a viscous Newtonian liquid. In this case

$$\boldsymbol{\sigma}^*=-p\boldsymbol{g}^*+2\mu\boldsymbol{D}^*, \tag{4.1}$$

where p is the pressure (deviation from the value p_∞ of the unperturbed hydrostatic pressure of the ambient air), μ is the viscosity, $\boldsymbol{g}^*$ is the metric tensor, $\boldsymbol{\sigma}^*$ is the stress tensor, and $\boldsymbol{D}^*$ is the strain rate tensor (the case of nonlinearly viscous (power-law) liquids is considered in the paper by Yarin (1982*a*)).

Using the gradient operator (2.3) and expression (3.8) for $\boldsymbol{v}$, we find the components of the strain rate tensor ($D_{ab} = \boldsymbol{a}\cdot\boldsymbol{D}^{*}\cdot\boldsymbol{b}$) in the form

$$\left.\begin{aligned}
D_{nn} &= \delta+\phi_{1,y}, \quad D_{nb} = D_{bn} = \tfrac{1}{2}(\phi_{2,y}+\phi_{1,z}),\\
D_{n\tau} &= D_{\tau n} = \tfrac{1}{2}(-\Omega_b+\lambda^{-1}V_{n,s}-\kappa V_b+kV_\tau\\
&\quad +\phi_{3,y}-z\lambda^{-1}\Omega_{\tau,s}+zk\Omega_n-yk\Omega_b+y\lambda^{-1}\delta_{,s}\\
&\quad +yk\lambda^{-1}V_{n,s}-yk\kappa V_b+yk^2V_\tau), \quad D_{bb} = \delta+\phi_{2,z},\\
D_{b\tau} &= D_{\tau b} = \tfrac{1}{2}(\Omega_n+\lambda^{-1}V_{b,s}+\kappa V_n+\phi_{3,z}\\
&\quad +y\lambda^{-1}\Omega_{\tau,s}+z\lambda^{-1}\delta_{,s}+yk\lambda^{-1}V_{b,s}+yk\kappa V_n),\\
D_{\tau\tau} &= \lambda^{-1}V_{\tau,s}-kV_n+z\lambda^{-1}\Omega_{n,s}-z\kappa\Omega_b+zk\Omega_\tau\\
&\quad -y\lambda^{-1}\Omega_{b,s}-y\kappa\Omega_n-\delta yk+yk\lambda^{-1}V_{\tau,s}-yk^2V_n.
\end{aligned}\right\} \tag{4.2}$$

From the incompressibility condition

$$\operatorname{tr}\boldsymbol{D}^{*} = 0, \tag{4.3}$$

$$\delta = -\tfrac{1}{2}(\lambda^{-1}V_{\tau,s}-kV_n), \tag{4.4}$$

$$\phi_{1,y}+\phi_{2,z} = z(k\Omega_b-k\Omega_\tau-\lambda^{-1}\Omega_{n,s})+y(\lambda^{-1}\Omega_{b,s}+\kappa\Omega_n+\delta k-k\lambda^{-1}V_{\tau,s}+k^2V_h). \tag{4.5}$$

Now we obtain expressions for the stresses in the jet using the constitutive equation (4.1). We note tentatively that at any rheology of liquid for closing the system of dynamic equations it is necessary to take into account the condition of absence (smallness) of stresses on the jet surface. It can readily be shown that this condition for the stresses inside the jet, gives estimates

$$\left.\begin{aligned}
&\sigma_{\tau n} = \sigma_{n\tau} = O(\epsilon\sigma_{\tau\tau}), \quad \sigma_{\tau b} = \sigma_{b\tau} = O(\epsilon\sigma_{\tau\tau}),\\
&\sigma_{nn} = O(\epsilon^2\sigma_{\tau\tau}), \quad \sigma_{bb} = O(\epsilon^2\sigma_{\tau\tau}), \quad \sigma_{bn} = \sigma_{nb} = O(\epsilon^2\sigma_{\tau\tau}).
\end{aligned}\right\} \tag{4.6}$$

The smallness of the stresses $\sigma_{n\tau}$ and $\sigma_{b\tau}$ makes it possible to show that the liquid cross-section remains flat in the linear approximation during motion, as assumed in the relations (3.8). On the other hand, after omitting the higher orders of small quantities, we have $\sigma_{nn} = \sigma_{bb}(=0)$. By virtue of this equality, the deformation of the liquid cross-section must be mainly isotropic, as assumed in the relations (3.8). It will be confirmed by straightforward calculation for a Newtonian liquid below. In addition, if we express the components of the stress tensor in terms of kinematic characteristics using the rheological equation of state of the liquid, then the estimates (4.6) will produce additional restrictions on the kinematics of motion.

Substituting (4.2) into (4.1) yields the components of the stress tensor in the form

$$\left.\begin{aligned}
\sigma_{nn} &= -p+2\mu(\delta+\phi_{1,y}), \quad \sigma_{bb} = -p+2\mu(\delta+\phi_{2,z}),\\
\sigma_{\tau\tau} &= -p+2\mu[\lambda^{-1}V_{\tau,s}-kV_n+z(\lambda^{-1}\Omega_{n,s}-\kappa\Omega_b+k\Omega_\tau)\\
&\quad +y(-\lambda^{-1}\Omega_{b,s}-\kappa\Omega_n-\delta k+k\lambda^{-1}V_{\tau,s}-k^2V_n)],\\
\sigma_{nb} &= \sigma_{bn} = \tfrac{1}{2}(\phi_{2,y}+\phi_{1,z}),\\
\sigma_{\tau n} &= \sigma_{n\tau} = \mu[-\Omega_b+\lambda^{-1}V_{n,s}-\kappa V_b+kV_\tau+z(k\Omega_n-\lambda^{-1}\Omega_{\tau,s})\\
&\quad +y(\lambda^{-1}\delta_{,s}-k\Omega_b+k\lambda^{-1}V_{n,s}-k\kappa V_b+k^2V_\tau)+\phi_{3,y}],\\
\sigma_{\tau b} &= \sigma_{b\tau} = \mu[\Omega_n+\lambda^{-1}V_{b,s}+\kappa V_n+z\lambda^{-1}\delta_{,s}+\phi_{3,z}\\
&\quad +y(\lambda^{-1}\Omega_{\tau,s}+k\lambda^{-1}V_{b,s}+k\kappa V_n)].
\end{aligned}\right\} \tag{4.7}$$

Using the estimates (4.6) for the stresses σ_{nn}, σ_{bb} and σ_{nb}, we have in the linear approximation

$$\left.\begin{aligned} -p+2\mu(\delta+\phi_{1,y}) &= 0, \\ -p+2\mu(\delta+\phi_{2,z}) &= 0, \\ \phi_{2,y}+\phi_{1,z} &= 0. \end{aligned}\right\} \tag{4.8}$$

From (4.8), taking into account (4.4) and (4.5), we have

$$\left.\begin{aligned} \phi_{1,y} = \phi_{2,z} &= \tfrac{1}{2}[z(\kappa\Omega_b - k\Omega_\tau - \lambda^{-1}\Omega_{n,s}) + y(\lambda^{-1}\Omega_{b,s} + \kappa\Omega_n - \tfrac{3}{2}k\lambda^{-1}V_{\tau,s} + \tfrac{3}{2}k^2 V_n)], \\ p &= 2\mu\delta + \mu[z(\kappa\Omega_b - k\Omega_\tau - \lambda^{-1}\Omega_{n,s}) + y(\lambda^{-1}\Omega_{b,s} + \kappa\Omega_n - \tfrac{3}{2}k\lambda^{-1}V_{\tau,s} + \tfrac{3}{2}k^2 V_n)]. \end{aligned}\right\} \tag{4.9}$$

Note that up to now we have not used the assumption that ϕ_1 and ϕ_2 are quadratic in y and z. From (4.8) and (4.9) it follows immediately that linear terms have the form $\phi_1 = Az$, $\phi_2 = -Ay$, $A = A(s)$. It means that the linear part of the field $\boldsymbol{v}_2$ corresponds to rigid rotation about the jet axis and can be included in Ω_τ. So the assumption about the quadratic form of ϕ_1 and ϕ_2 is justified, and the lateral deformation of the jet cross-section is mainly isotropic.

The estimates (4.6) for the stresses $\sigma_{\tau n}$ and $\sigma_{\tau b}$, taking into account (4.7), give additional kinematic conditions

$$\Omega_n = -\lambda^{-1}V_{b,s} - \kappa V_n, \quad \Omega_b = \lambda^{-1}V_{n,s} - \kappa V_b + kV_\tau, \tag{4.10}$$

used subsequently for closing the system of equations of the problem. In this case the expressions for the axial values of the stresses and longitudinal force are of the form

$$\Sigma_{\tau\tau} = 3\mu(\lambda^{-1}V_{\tau,s} - kV_n), \quad \Sigma_{\tau n} = \Sigma_{\tau b} = 0, \quad P = f\Sigma_{\tau\tau}. \tag{4.11}$$

Note that the need for the relations (4.10) arises from the fact that the shearing force $\boldsymbol{Q}$ in the cross-section is not determined from the rheological relations of the given accuracy. In fact, in virtue of the estimates (4.6) and expressions (4.7) for $\sigma_{\tau n}$ and $\sigma_{\tau b}$, $|\boldsymbol{Q}|$ turns out to be $O(\epsilon^2 P)$ because the terms in (4.7), linear with respect to y and z, provide no contribution to the shearing force. Consequently, from the rheological relations of the given accuracy it is impossible to derive an explicit expression for the shearing force and it can only be determined from the solution of the problem.

The fact that $|\boldsymbol{Q}| = O(\epsilon^2 P)$ enables one to neglect the term associated with the shearing force in the projection of the momentum equation onto the tangent to the jet axis. In projecting the momentum equation onto the normal the terms associated with the longitudinal and shearing forces are of the same order of magnitude due to the smallness of curvature of the jet axis in the problem at hand.

Using (4.8) and (4.9), we can calculate the quadratic functions ϕ_1 and ϕ_2 explicitly:

$$\left.\begin{aligned} \phi_1 &= \tfrac{1}{2}(\kappa\Omega_b - k\Omega_\tau - \lambda^{-1}\Omega_{n,s})\,yz \\ &\quad + \tfrac{1}{4}(\lambda^{-1}\Omega_{b,s} + \kappa\Omega_n - \tfrac{3}{2}k\lambda^{-1}V_{\tau,s} + \tfrac{3}{2}k^2 V_n)(y^2 - z^2) + O(\epsilon^3|\boldsymbol{V}|), \\ \phi_2 &= \tfrac{1}{4}(\kappa\Omega_b - k\Omega_\tau - \lambda^{-1}\Omega_{n,s})(z^2 - y^2) \\ &\quad + \tfrac{1}{2}(\lambda^{-1}\Omega_{b,s} + \kappa\Omega_n - \tfrac{3}{2}k\lambda^{-1}V_{\tau,s} + \tfrac{3}{2}k^2 V_n)\,yz + O(\epsilon^3|\boldsymbol{V}|). \end{aligned}\right\} \tag{4.12}$$

From (3.13) and (4.7), taking into account the expression for the pressure (4.9), we find expressions of the projections of the stress moment in the cross-section in terms of kinematic characteristics. More specifically, for the cross-section possessing double symmetry ($I_{bn} = 0$, $I_n = I_b = I$) we have

$$\left.\begin{aligned} M_n &= 3\mu I(\lambda^{-1}\Omega_{n,s}+k\Omega_\tau-\kappa\Omega_b), \\ M_b &= 3\mu I(\lambda^{-1}\Omega_{b,s}+\kappa\Omega_n-\tfrac{3}{2}k\lambda^{-1}V_{\tau,s}+\tfrac{3}{2}k^2V_n), \\ M_\tau &= \mu I(2\lambda^{-1}\Omega_{\tau,s}+k\lambda^{-1}V_{b,s}+k\kappa V_n-k\Omega_n). \end{aligned}\right\} \quad (4.13)$$

For this cross-section, in virtue of (4.12), the terms appearing in the moment-of-momentum equation and associated with the integral $\boldsymbol{j}_1$ are equal to zero with the required accuracy.

It is easy to see that the system of equations of the jet dynamics (3.10)–(3.12), taking into account the kinematic and geometric relations of §2 and (3.13)–(3.16), (4.10), (4.11) and (4.13), turns out to be closed in this case. If the cross-section does not possess the required symmetry the problem remains unclosed insofar as the term j_1, has been expressed through the function ϕ_3 that remained indefinite, is significant. (Note that the function ϕ_3 can be determined explicitly only in the case of a jet of circular cross-section, but precisely in this case there is no need for the explicit expression for ϕ_3 because of the double symmetry.)

It is worth noting that the inclusion of the contribution of surface tension and normal stresses on the lateral surface (air pressure) leads to the following expressions for the stresses in the liquid inside the jet (in that case it is natural to consider only jets of circular cross-section):

$$\left.\begin{aligned} \sigma_{nn} &= \sigma_{bb} = -\alpha[G-ka^{-1}y(1+\lambda^{-2}a_{,s}^2)^{-\frac{3}{2}}]-p_1', \\ G &= a^{-1}(1+\lambda^{-2}a_{,s}^2)^{-\frac{1}{2}}-(1+\lambda^{-2}a_{,s}^2)^{-\frac{3}{2}}\lambda^{-1}[\lambda^{-1}a_{,s}]_{,s} \end{aligned}\right\} \quad (4.14)$$

(previously the contribution of air pressure was specified only by the distributed force and moment $\boldsymbol{q}$ and $\boldsymbol{m}$).

Here $a = a(s,t)$ is the radius of the jet, α is the surface-tension coefficient, G is the double mean curvature of the jet surface and p_1' is the disturbance of air pressure on the jet surface with respect to p_∞. The pressure p_1' as a function of the jet-axis configuration will be defined in §5.

One can easily see that the assumption that the liquid cross-section remains flat during motion is still valid and the deformation of the liquid cross-section is mainly isotropic. With the aid of the first equations in (4.7) and (4.9), using (4.14), we find the distribution of the pressure in the jet

$$\begin{aligned} p = 2\mu\delta+\mu[z(\kappa\Omega_b-k\Omega_\tau-\lambda^{-1}\Omega_{n,s})+y(\lambda^{-1}\Omega_{b,s}+\kappa\Omega_n-\tfrac{3}{2}k\lambda^{-1}V_{\tau,s}+\tfrac{3}{2}k^2V_n)] \\ +\alpha[G-ka^{-1}y(1+\lambda^{-2}a_{,s}^2)^{-\frac{3}{2}}]+p_1' \end{aligned} \quad (4.15)$$

(cf. the last equation in (4.9)). Hence, with the aid of (3.13) and (4.7), we find new expressions for P, M_n and M_b:

$$\left.\begin{aligned} P &= \Sigma_{\tau\tau}'f-\int_{\mathscr{D}}p_1'\,\mathrm{d}S+P_\alpha, \quad \Sigma_{\tau\tau}' = 3\mu(\lambda^{-1}V_{\tau,s}-kV_n)-\alpha G, \\ P_\alpha &= 2\pi a\alpha(1+\lambda^{-2}a_{,s}^2)^{-\frac{1}{2}}, \\ M_n &= 3\mu I(\lambda^{-1}\Omega_{n,s}+\kappa\Omega_\tau-\kappa\Omega_b)-\int_{\mathscr{D}}zp_1'\,\mathrm{d}S, \\ M_b &= 3\mu I(\lambda^{-1}\Omega_{b,s}+\kappa\Omega_n-\tfrac{3}{2}k\lambda^{-1}V_{\tau,s}+\tfrac{3}{2}k^2V_n) \\ &\quad -\alpha kIa^{-1}(1+\lambda^{-2}a_{,s}^2)^{-\frac{3}{2}}+\int_{\mathscr{D}}yp_1'\,\mathrm{d}S. \end{aligned}\right\} \quad (4.16)$$

All the other equations and relations of the one-dimensional theory of jets remain unchanged; the system of equations is still closed.

The closed system of equations thus obtained takes into account the stress moment and shearing force in the jet cross-section. For very thin jets the influence of the moment and shearing force can be neglected. As a result we obtain the closed system of equations of momentless theory. For example, if the jet axis is a curve lying in the (ξ, η)-plane then the equations of the momentless theory of bending of jets with the circular cross-section have the form

$$\left.\begin{aligned}
&\frac{\partial \lambda f}{\partial t}+\frac{\partial f W}{\partial s}=0, \quad f=\pi a^2,\\
&\frac{\partial \lambda f V_\tau}{\partial t}-\frac{f V_n}{\lambda}\frac{\partial \lambda V_n}{\partial s}+\frac{\partial f V_\tau W}{\partial s}-\lambda f W k V_n=\frac{1}{\rho}\frac{\partial P}{\partial s}+\lambda f F_\tau+\frac{1}{\rho}\lambda q_\tau,\\
&\frac{\partial \lambda f V_n}{\partial t}+\frac{f V_\tau}{\lambda}\frac{\partial \lambda V_n}{\partial s}+\frac{\partial f V_n W}{\partial s}+\lambda k f V_\tau W=\frac{1}{\rho}\lambda k P+\lambda f F_n+\frac{1}{\rho}\lambda q_n,\\
&k=\frac{\partial^2 H}{\partial s^2}\left[1+\left(\frac{\partial H}{\partial s}\right)^2\right]^{-\frac{3}{2}}, \quad \frac{\partial H}{\partial t}=\lambda V_n, \quad \lambda=\left[1+\left(\frac{\partial H}{\partial s}\right)^2\right]^{\frac{1}{2}}, \quad W=V_\tau-V_n\frac{\partial H}{\partial s}.
\end{aligned}\right\} \tag{4.17}$$

The expression for the longitudinal force P is given by (4.16).

5. The aerodynamic forces acting on the curved jet

We calculate the distributed force $\boldsymbol{q}$ and moment $\boldsymbol{m}$ applied to the jet axis. For this purpose we use the theory of motion of slender 'fishlike' bodies (Lighthill 1960; Wu 1971; Logvinovitch 1973). Let us introduce the Cartesian system of coordinates $O_1\xi\eta\zeta$ whose axis $O_1\xi$ coincides with the axis of an unperturbed jet and moves together with it with a velocity U_0 in the direction $\xi = -\infty$. We parametrize the jet axis in the same way as in §2, $\xi = s$, and write its equations in the form

$$\eta = H(s,t), \quad \zeta = Z(s,t), \tag{5.1}$$

where H and Z are the displacements of the axis in the directions $O_1\eta$ and $O_1\zeta$.

First consider small three-dimensional disturbances when H and Z are of first order, the higher orders of small quantities being neglected. We also neglect the change of the jet radius in the process of growth of small bending disturbances (we substantiate this in §6). The gas surrounding the jet will be considered perfect and incompressible and its motion will be taken to be potential.

We then find the pressure distribution on the jet surface to be

$$p_1 = p_\infty + p_1' = p_\infty + \rho_1[(\eta-H)\,\mathrm{D}V_\eta^* + (\zeta-Z)\,\mathrm{D}V_\zeta^*], \tag{5.2}$$

$$\mathrm{D} = \frac{\partial}{\partial t}+U_0\frac{\partial}{\partial s}, \quad V_\eta^* = \mathrm{D}H, \quad V_\zeta^* = \mathrm{D}Z$$

(p_∞ is the pressure at infinity and ρ_1 is the air density); the linear density of forces applied to the jet is

$$\boldsymbol{q} = -\rho_1 U_0^2 f_0(\boldsymbol{j}H_{,ss}+\boldsymbol{k}Z_{,ss}), \quad f_0 = \pi a_0^2. \tag{5.3}$$

Note that in virtue of the inequality $\rho_1 \ll \rho$ only the contribution of terms of order $\rho_1 U_0^2$ will be significant in determining the dynamic action of air on the jet.

Obviously, in the given case the linear density of the moment of external forces $\boldsymbol{m}$ is zero.

We now generalize the result obtained to the case of finite disturbances of the jet axis. In this case we consider only the two-dimensional bending. Since only the terms of order $\rho_1 U_0^2$ are of interest in calculating the aerodynamic force $\boldsymbol{q}$ we can consider the jet as stationary at each instant. The calculations have been carried out using the coordinate system adopted in §2 and associated with the curved jet axis. The linear density of aerodynamic force in the case of finite two-dimensional disturbances of the jet axis is

$$\boldsymbol{q} = -\rho_1 U_0^2 f H_{,ss}(1+H_{,s}^2)^{-\frac{3}{2}}\boldsymbol{n}. \tag{5.4}$$

In this case, as formerly, $\boldsymbol{m} = 0$.

In conclusion of this section we note that the aerodynamic force calculated from the theory of motion of slender bodies (5.4) adequately describes the action of air flow on the jet only in the case of sufficiently small bending disturbances when the influence of the boundary layer in air is insignificant. With increasing disturbances their development begins to be significantly affected by the drag caused by the boundary-layer separation at the bends of the jet (in the case of sufficiently viscous liquids the air friction drag seems to be unimportant for the jet). The drag force deforms the bending disturbances of the jet, shaping them like a succession of breaking waves. A rigorous calculation of the drag distributed along the jet is impossible at present. However, qualitative information on the effect of drag on the development of bending disturbances of the jet can be obtained by using the empirical drag coefficient for the transverse flow past a cylinder.

The velocity of the flow incident on the jet along the normal as $y \to \infty$ is

$$U_\infty = -U_0 H_{,s}(1+H_{,s}^2)^{-\frac{1}{2}} \tag{5.5}$$

(as formerly, the two-dimensional bending of the jet is considered).

Considering the jet of circular section and putting the drag coefficient $C_D = 1$ (see Goldstein 1938), we have for the drag the expression

$$\boldsymbol{q}_1 = -\rho_1 U_0^2 a H_{,s}^2 (1+H_{,s}^2)^{-1} \operatorname{sgn}(H_{,s})\boldsymbol{n}. \tag{5.6}$$

Thus the total aerodynamic force acting on the perturbed jet is equal to the sum of the forces determined by (5.4) and (5.6). The drag (5.6) is quadratic in the amplitude of disturbances of the jet axis so that it need not be taken into account in the linear analysis to which we turn our attention now.

6. The transverse stability of the liquid jet in the air stream

In this section we investigate the stability of a straight laminar jet of the Newtonian viscous liquid with respect to small long-wave three-dimensional bending disturbances. In this case the equations of the dynamics of thin liquid jets obtained above take the form

$$\left.\begin{aligned}
&\frac{\partial f}{\partial t}+f_0\frac{\partial V_\tau}{\partial s} = 0, \quad \rho f_0\frac{\partial \boldsymbol{V}}{\partial t} = \frac{\partial}{\partial s}(P\boldsymbol{\tau}+\boldsymbol{Q})+\boldsymbol{q}, \quad \rho\frac{\partial \boldsymbol{K}}{\partial t} = \frac{\partial \boldsymbol{M}}{\partial s}+\boldsymbol{\tau}\times\boldsymbol{Q},\\
&\boldsymbol{K} = I(\boldsymbol{n}\Omega_n+\boldsymbol{b}\Omega_b), \quad \Omega_n = -V_{b,s}-\kappa V_n, \quad \Omega_b = V_{n,s}-\kappa V_b,\\
&\boldsymbol{M} = 3\mu I[\boldsymbol{n}(\Omega_{n,s}-\kappa\Omega_b)+\boldsymbol{b}(\Omega_{b,s}+\kappa\Omega_n)]-\alpha I a_0^{-1} k\boldsymbol{b}+\boldsymbol{M}_1,\\
&P = 3\mu f_0\frac{\partial V_\tau}{\partial s}+\pi a\alpha+\alpha f_0\frac{\partial^2 a}{\partial s^2}+P_1, \quad f_0 = \pi a_0^2, \quad I = \tfrac{1}{4}\pi a_0^4.
\end{aligned}\right\} \tag{6.1}$$

Here a_0 is the radius of the unperturbed jet. Equations (6.1) are obtained by linearizing the relations (3.10)–(3.12), (3.14), (4.10), (4.13) and (4.16), noting that in the adopted frame of reference the unperturbed jet is at rest while the air moves along its axis with the velocity U_0. It is also assumed that the normal cross-section of the jet is circular, and the mass forces and rotation of the liquid about the jet axis are absent ($\Omega_\tau = 0$). In the moment-of-momentum equation there is no linear density of the moment of external forces $\boldsymbol{m}$, since subsequently we consider the case where $\boldsymbol{m} = 0$. $\boldsymbol{M}_1$ and P_1 in the expressions for the moment of stresses and the value of the longitudinal force in the cross-section represent the terms from (4.16), which depend upon disturbances of the ambient air pressure, in which case it is clear that $P_1 \sim \rho_1 U_0^2 a_{,ss}$.

We calculate the moment $\boldsymbol{M}_1$ required for investigating the bending stability. The Cartesian coordinates introduced in §§2 and 5 are related to the axes which are determined by the moving trihedron associated with the jet axis by the relations

$$\left.\begin{aligned} \eta - H &= yn_\eta + zb_\eta, \\ \zeta - Z &= yn_\zeta + zb_\zeta. \end{aligned}\right\} \tag{6.2}$$

Here the subscripts η and ζ denote the projections of the unit normal and binormal vectors onto the axes $O_1\eta$ and $O_1\zeta$ respectively. The latter relations lead in the linear approximation to the equation

$$(\eta - H) H_{,ss} + (\zeta - Z) Z_{,ss} = ky. \tag{6.3}$$

Retaining only the terms of order $\rho_1 U_0^2$ in the expression (5.2) for p_1', we find

$$p_1' = \rho_1 U_0^2 ky, \tag{6.4}$$

which, taking into account (4.16), enables us to calculate the moment

$$\boldsymbol{M}_1 = \rho_1 U_0^2 kI\boldsymbol{b}. \tag{6.5}$$

Projecting the momentum and moment-of-momentum equations (6.1) onto the normal, binormal and tangent to the jet axis and taking into account the smallness of the components of the velocity $\boldsymbol{V}$ and the angular velocity $\boldsymbol{\Omega}$ of the liquid, we find

$$\left.\begin{aligned} &2\frac{\partial a}{\partial t} + a_0 \frac{\partial V_\tau}{\partial s} = 0, \quad \rho f_0 \frac{\partial V_\tau}{\partial t} = 3\mu f_0 \frac{\partial^2 V_\tau}{\partial s^2} + \pi\alpha \frac{\partial a}{\partial s} + \alpha f_0 \frac{\partial^3 a}{\partial s^3} + \frac{\partial P_1}{\partial s} + q_\tau, \\ &\rho f_0 \frac{\partial V_n}{\partial t} = \frac{\partial Q_n}{\partial s} - \kappa Q_b + \pi a_0 k\alpha + q_n, \quad \rho f_0 \frac{\partial V_b}{\partial t} = \frac{\partial Q_b}{\partial s} + \kappa Q_n + q_b, \\ &-\rho I \frac{\partial^2 V_b}{\partial s\,\partial t} - \rho I \frac{\partial}{\partial t}(\kappa V_n) = \frac{\partial M_n}{\partial s} - \kappa M_b - Q_b, \\ &\rho I \frac{\partial^2 V_n}{\partial s\,\partial t} - \rho I \frac{\partial}{\partial t}(\kappa V_b) = \frac{\partial M_b}{\partial s} + \kappa M_n + Q_n. \end{aligned}\right\} \tag{6.6}$$

Here only the small quantities of the first order are retained; the projection of the moment-of-momentum equation onto the tangent being identically equal to zero.

The first two equations of the system (6.6), which describe the growth of small axially symmetric disturbances of the jet, can be solved independently of the remaining ones (the growth rate of axially symmetric disturbances γ_1 was found by Weber 1931). On the other hand, the remaining equations (6.6) describe small bending disturbances of the liquid jet with neglect of the change in its radius. These disturbances have a growth rate γ for which the corresponding characteristic equation has been derived in this section. If the maximum value of γ is much greater than that of γ_1, the bending disturbances grow much more rapidly than the axially symmetric ones; in this case it may be assumed that the jet radius remains unchanged.

We have (5.3) as an expression for the aerodynamic force acting on the jet, the linear density of the moment of external forces is zero. Furthermore, the last four equations of (6.6), which describe small bending disturbances, are supplemented with expressions, derived from (6.1) and (6.5), for the components of the moment of stresses in the jet cross-section

$$\left.\begin{aligned} M_n &= 3\mu I\left[-\frac{\partial^2 V_b}{\partial s^2}-\frac{\partial}{\partial s}(\kappa V_n)-\kappa\frac{\partial V_n}{\partial s}+\kappa^2 V_b\right], \\ M_b &= 3\mu I\left[\frac{\partial^2 V_n}{\partial s^2}-\frac{\partial}{\partial s}(\kappa V_b)-\kappa\frac{\partial V_b}{\partial s}-\kappa^2 V_n\right]-\frac{\alpha}{a_0}Ik\left(1-\frac{\rho_1 a_0 U_0^2}{\alpha}\right). \end{aligned}\right\} \tag{6.7}$$

By virtue of the linearity of the problem for small bending disturbances it suffices to consider the helical disturbance of the jet axis

$$\left.\begin{aligned} H = A\exp(\gamma t)\cos\frac{\chi s}{a_0}, \quad Z &= B\exp(\gamma t)\sin\frac{\chi s}{a_0}, \\ \chi &= \frac{2\pi a_0}{l}, \end{aligned}\right\} \tag{6.8}$$

where l is the disturbance wavelength. Hence the curvature and torsion are determined by the relations

$$\left.\begin{aligned} k &= \left(\frac{\chi}{a_0}\right)^2\exp(\gamma t)\left[A^2\cos^2\frac{\chi s}{a_0}+B^2\sin^2\frac{\chi s}{a_0}\right]^{\frac{1}{2}}, \\ \kappa &= \chi AB a_0^{-1}\left[A^2\cos^2\frac{\chi s}{a_0}+B^2\sin^2\frac{\chi s}{a_0}\right]^{-1}. \end{aligned}\right\} \tag{6.9}$$

Using the kinematic relation (2.6), which relates the velocity $\boldsymbol{U}$ of the point s of the jet axis to the velocity $\boldsymbol{V}$ of the liquid particle located at this point,

$$\boldsymbol{U} = \boldsymbol{V}-\boldsymbol{\tau}(\boldsymbol{V}\cdot\boldsymbol{i}), \quad \boldsymbol{U} = \boldsymbol{j}H_{,t}+\boldsymbol{k}Z_{,t}, \tag{6.10}$$

we obtain, taking into account (6.8),

$$\left.\begin{aligned} V_n &= -\gamma\exp(\gamma t)\left[A^2\cos^2\frac{\chi s}{a_0}+B^2\sin^2\frac{\chi s}{a_0}\right]^{\frac{1}{2}}, \\ V_b &= 0. \end{aligned}\right\} \tag{6.11}$$

On projecting the relation (5.3) for $\boldsymbol{q}$ onto the normal and binormal to the jet axis, we find with the aid of (6.8) to the required accuracy

$$\left.\begin{aligned} q_n &= -\rho_1 U_0^2 f_0 \chi^2 a_0^{-2}\exp(\gamma t)\left[A^2\cos^2\frac{\chi s}{a_0}+B^2\sin^2\frac{\chi s}{a_0}\right]^{\frac{1}{2}}, \\ q_b &= 0, \end{aligned}\right\} \tag{6.12}$$

Substituting (6.9), (6.11) and (6.12) into the last four relations of (6.6) and (6.7) yields equations for small three-dimensional disturbances of the thin jet:

$$\left.\begin{aligned} &-\rho f_0\frac{\partial V_n}{\partial t}+\frac{\partial Q_n}{\partial s}-\kappa Q_b+\pi a_0\alpha k+q_n=0, \quad \frac{\partial Q_b}{\partial s}+\kappa Q_n=0,\\ &-\rho I\frac{\partial^2 V_n}{\partial s\,\partial t}+\frac{\partial M_b}{\partial s}+Q_n=0, \quad \rho I\kappa\frac{\partial V_n}{\partial t}-\kappa M_b-Q_b=0,\\ &\frac{\partial^2 V_n}{\partial s^2}=\frac{M_b}{3\mu I}+\kappa^2 V_n+\frac{\alpha k}{3\mu a_0}\left(1-\frac{\rho_1 a_0 U_0^2}{\alpha}\right). \end{aligned}\right\} \tag{6.13}$$

In this case

$$M_n=0. \tag{6.14}$$

Note that the equations of two-dimensional bending of the jet follow from (6.13) when $\kappa=Q_b=0$.

On finding the M_b from the last equation of (6.13), we obtain with the aid of the last but one equation of (6.13) the projection of the shearing force onto the binormal:

$$Q_b=\rho\kappa I\frac{\partial V_n}{\partial t}-3\mu\kappa I\left(\frac{\partial^2 V_n}{\partial s^2}-\kappa^2 V_n\right)+\frac{\alpha\kappa k I}{a_0}\left(1-\frac{\rho_1 a_0 U_0^2}{\alpha}\right). \tag{6.15}$$

Then the third equation of (6.13) gives

$$Q_n=\rho I\frac{\partial^2 V_n}{\partial s\,\partial t}-3\mu I\left[\frac{\partial^3 V_n}{\partial s^3}-\frac{\partial}{\partial s}(\kappa^2 V_n)\right]+\frac{\alpha I}{a_0}\left(1-\frac{\rho_1 a_0 U_0^2}{\alpha}\right)\frac{\partial k}{\partial s}. \tag{6.16}$$

Substituting (6.15) and (6.16) into the second equation of (6.13) leads, taking into account (6.9) and (6.11), to an identity, while the first equation of (6.13), with the use of (6.15) and (6.16), yields

$$\begin{aligned} -\rho f_0\frac{\partial V_n}{\partial t}+\rho I\left(\frac{\partial^3 V_n}{\partial s^2\,\partial t}-\kappa^2\frac{\partial V_n}{\partial t}\right)+3\mu I\left[\kappa^2\left(\frac{\partial^2 V_n}{\partial s^2}-\kappa^2 V_n\right)\frac{\partial^4 V_n}{\partial s^4}+\frac{\partial^2}{\partial s^2}(\kappa^2 V_n)\right]=0.\\ +\frac{\alpha I}{a_0}\left(1-\frac{\rho_1 a_0 U_0^2}{\alpha}\right)\left(\frac{\partial^2 k}{\partial s^2}-k\kappa^2\right)+\pi\alpha a_0 k+q_n \end{aligned} \tag{6.17}$$

The relation thus obtained is reduced with the use of (6.9), (6.11) and (6.12) to the equation

$$\begin{aligned} &\left[A^2\cos^2\frac{\chi s}{a_0}+B^2\sin^2\frac{\chi s}{a_0}\right]^2\left[\gamma^2\left(\frac{\rho f_0 a_0}{\chi}+\frac{\rho I\chi}{a_0}\right)\right.\\ &\qquad\left.+\frac{3\mu\chi^3 I}{a_0^3}\gamma+\pi\alpha\chi-\rho_1 U_0^2\pi a_0\chi-\frac{\alpha I\chi^3}{a_0^4}\left(1-\frac{\rho_1 a_0 U_0^2}{\alpha}\right)\right]=0. \end{aligned} \tag{6.18}$$

We thus obtain the characteristic equation for small bending disturbances:

$$\gamma^2+\frac{3\mu\chi^4}{4\rho a_0^2}\gamma+\left(\frac{\alpha}{\rho a_0^3}-\frac{\rho_1 U_0^2}{\rho a_0^2}\right)\chi^2=0. \tag{6.19}$$

Here only the leading terms for small χ are retained, because the long-wave approximation is considered.

The characteristic equation (6.19) determines the growth rate of both the two- and the three-dimensional disturbances of the jet axis. Therefore, in the cases of both the two- and three-dimensional disturbances at the jet velocities

$$U_0 > U_0^* = \left(\frac{\alpha}{\rho_1 a_0}\right)^{\frac{1}{2}}, \tag{6.20}$$

(6.19) predicts instability (there is a real root $\gamma > 0$), since the surface-tension forces fail to oppose the dynamic action of the air. The condition (6.20) had been previously obtained by Debye & Daen (1959) for the two-dimensional bending of the inviscid liquid jet (see also Weber 1931). Thus the critical velocity value corresponding to the onset of bending instability is the same for two- and three-dimensional disturbances and is independent of viscosity. It is worth noting that the characteristic equation (6.19) may be obtained by means of the energy balance for small one-mode bending disturbance of the jet in the air stream (Entov & Yarin 1979; Yarin 1982*a*).

An important circumstance is the fact that the two- and three-dimensional bending disturbances of the jet grow with the same rate, and consequently the disintegration of the jet must be of a helical character. Photographs of the jet in two projections (Ivanov 1966) support this conclusion.

When the inequality (6.20) holds, using (6.19) we find the dimensionless wavenumber χ_* of the most rapidly growing disturbance and the maximum growth rate γ_*:

$$\chi_* = \left[\frac{8}{9}\frac{\rho a_0^2}{\mu^2}\left(\rho_1 U_0^2 - \frac{\alpha}{a_0}\right)\right]^{\frac{1}{6}}, \quad \gamma_* = \frac{(a_0 \rho_1 U_0^2 - \alpha)^{\frac{2}{3}}}{(3\mu\rho a_0^4)^{\frac{1}{3}}}. \tag{6.21}$$

If the surface tension is insignificant and the liquid is sufficiently viscous,

$$\rho_1 U_0^2 \gg \frac{\alpha}{a_0}, \quad \mu^2(\rho a_0^2 \rho_1 U_0^2)^{-1} \gg 1, \tag{6.22}$$

then the growth rate of axisymmetric disturbances γ_{1*} is small in comparison with γ_*. In this case it is possible to neglect the change of jet radius, and bending disturbances become predominant.

Note that the phenomenon of instability of jets with respect to bending disturbances was discovered by Haenlein (1932) in experiments with castor-oil jets with the parameters satisfying (6.22).

Assuming that the breakup of the jet effluent from a nozzle with a velocity U_0 is due to the most rapidly growing disturbance and occurs when the maximum deviation of the jet axis is of order na_0 ($n = 2$–4), we find the breakup length of the jet

$$L = \Delta\left[\frac{3\mu\rho a_0^2 U_0^3}{(\rho_1 U_0^2 - \alpha/a_0)^2}\right]^{\frac{1}{3}}. \tag{6.23}$$

Here $\Delta = \ln(na_0/\delta_0)$, and δ_0 is the maximum deviation of the jet axis from the straight line when $t = 0$. The arguments in favour of the chosen break-up condition will be outlined in §8. As usual in the theory of jet disintegration the quality Δ is virtually an empirical constant because of the uncertainty of the value of δ_0.

In conclusion we note that there is every reason to believe that neglect of the viscosity of the air surrounding the jet leads to overestimated values of disturbances of the pressure on the jet surface (Ivanov 1966; Sterling & Sleicher 1975). It appears that the viscosity smoothes out the streamlines calculated for the potential flow past the jet and thereby decreases the pressure disturbances. This circumstance can be taken into account by analogy with the paper by Sterling & Sleicher (1975) by modifying the expression (5.2) for the pressure p_1':

$$p_1' = C\rho_1[(\eta - H)\,\mathrm{D}V_\eta^* + (\zeta - Z)\,\mathrm{D}V_\zeta^*], \tag{6.24}$$

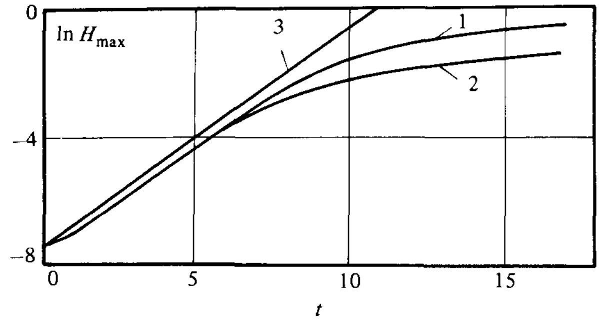

FIGURE 2. Changes in the disturbance amplitude with time in the absence of the drag. (1) for the liquid with the viscosity $\mu = 10$ P; (2) $\mu = 100$ P (1 and 2 have been obtained by integrating the nonlinear equations); (3) the result of the linear theory.

where C is an empirical constant and $0 < C < 1$. In this case in all expressions of this section ρ_1 will be replaced by the product $C\rho_1$, which will lead, in particular, to an increase in the threshold velocity U_0^*.

In the linear analysis of the rate of growth of small bending disturbances, an important stabilizing factor is ignored: viscous stresses arising from extension of the jet axis. The influence of this and other nonlinear effects on the rate of growth of bending disturbances can be investigated in the numerical solution of one-dimensional equations of the dynamics of liquid jets, to which §7 is devoted.

7. The numerical investigation of instability of thin liquid jets

Here we outline some results of the numerical investigation into the dynamics of finite two-dimensional bending disturbances of a viscous liquid jet having a circular cross-section. First of all we discuss the variant of the problem with neglect of the drag. In this case the disturbance of an infinite, initially straight jet appears as a standing wave with the amplitude increasing in time. For a sufficiently viscous liquid one can neglect the inertial terms in comparison with the viscous ones in all equations of the problem except for the projection of the momentum equation onto the normal to the jet axis (the viscous terms in this equation are small: they have the order of the shearing force). For details of the indicated transformations see Entov & Yarin (1979) and Yarin (1982*b*).

In calculations the initial disturbance had prescribed values of

$$V_n = V_\tau = 0, \quad a = a_0, \quad H = H_0 \sin\frac{2\pi s}{l}, \quad H_0 = (5\times10^{-4}\text{–}5\times10^{-2})\,l. \qquad (7.1)$$

The boundary conditions were obtained from the conditions of periodicity. The parameter s is a distance measured along the axis of the unperturbed jet $O_1\xi$, l being the disturbance wavelength.

The numerical realization of the obtained system of one-dimensional equations was effected with the aid of an implicit finite-difference scheme whose spectrum for small disturbances reproduced well the spectrum of the linearized differential problem.

In calculations we investigated the development of bending disturbances of jets of highly viscous liquids ($\mu = 10\text{–}10^3$ P, $\rho = 1$ g/cm^3, $a_0 = 10^{-1}$ cm) moving in a low-density medium ($\rho_1 = 10^{-3}$ g/cm^3) with a velocity $U_0 = 10^3$ cm/s. It is worth noting that in the bending of jets of highly viscous liquids the surface tension is negligible.

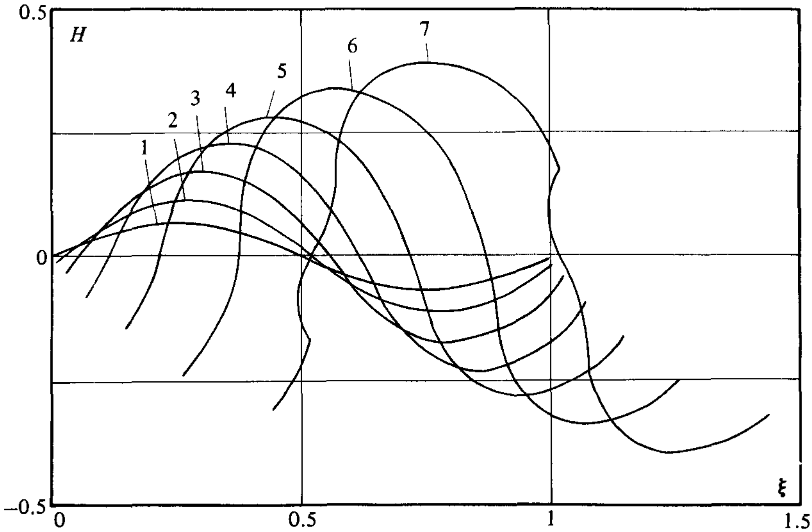

FIGURE 3. Configurations of the jet axis segment corresponding to one disturbance wave (μ = 10 P, the drag being taken into account). The dimensionless time is indicated with numbers by the curves.

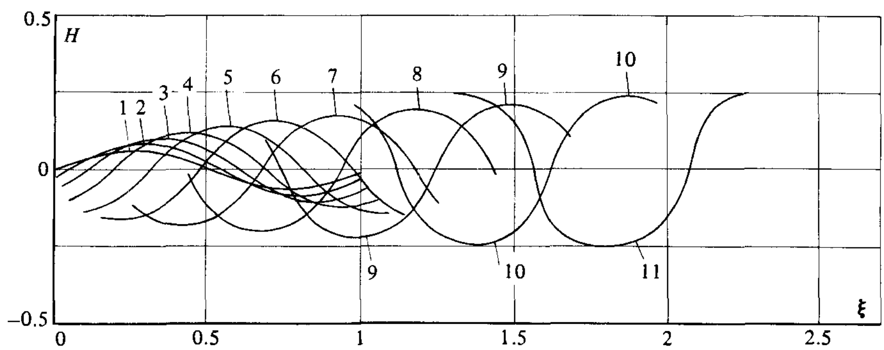

FIGURE 4. Configurations of the jet axis segment corresponding to one disturbance wave (μ = 100 P, the drag being taken into account). The dimensionless time is indicated with numbers by the curves.

The data obtained with neglect of the drag indicate that the small initial disturbance of the jet of the form (7.1) with $H_0 = 5 \times 10^{-4} l$ grows with the rate predicted by the linear theory outlined in §6. This is evidenced by comparison in figure 2 of the slopes of the straight regions of curves 1 (μ = 10 P) and 2 (μ = 100 P) with straight line 3 corresponding to the linear theory. With a further increase in the amplitude of the disturbance its harmonic shape is distorted and the growth rate decreases. The latter takes place under the action of viscous stresses which are due to the nonlinear effect, namely the extension of the jet axis in bending. Here and elsewhere in the figures the time values are referred to $T = (\rho\mu a_0^2/\rho_1^2 U_0^4)^{\frac{1}{3}}$, which is the timescale of growth of bending disturbances according to the linear theory. As a linear scale serves the wavelength of the disturbance most rapidly growing according to the linear theory: $l = 2\pi(9\mu^2 a_0^4/8\rho\rho_1 U_0^2)^{\frac{1}{6}}$. In the case of μ = 10 P the scales are: $T = 0.0047$ s, $l = 0.943$ cm and in the case of μ = 100 P $T = 0.01$ s and $l = 2.02$ cm. Data are given which relate to disturbances with the wavelength to which the maximum growth rate at the linear stage corresponds.

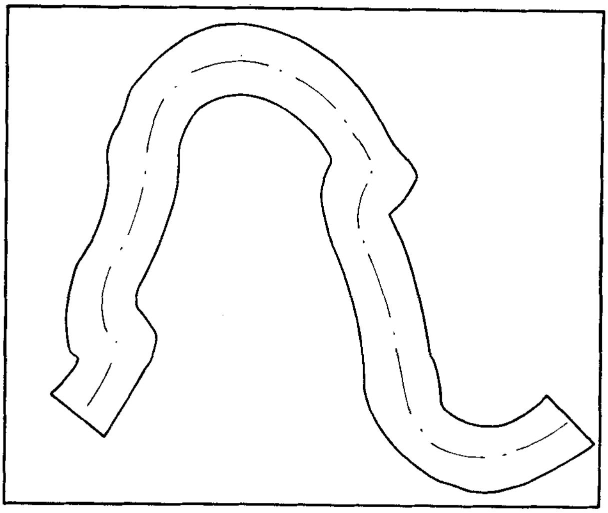

FIGURE 5. The form of the jet segment at the moment of 'overturning'. The dimensionless time is 7 (0.0329 s), the radius ranging from 65 % to 80 % of the initial value.

In the presence of the drag and sufficiently high viscosity of the liquid, the jet motion can be distinctly divided into two components of different nature. One of them is the deformation of the jet under the action of the 'lift' component of the aerodynamic force, and the other is a drift of the jet disturbance as a whole in the direction of the action of the drag. As a result of the drift, the jet axis can take a rather complicated shape and 'overturns' occur; therefore we have to discard the parametrization of the jet axis described in §2 in favour of the Lagrangian parametrization for which $\mathrm{d}s/\mathrm{d}t = 0$.

In calculations with the inclusion of the drag, the evolution of one half of the disturbance wave was investigated. Neglect of the inertial terms, construction of the finite-difference scheme and the values of the parameters were the same as in the variant without the drag.

In calculations neglecting the drag, the bending disturbances represent a system of standing waves with the amplitude increasing in time while the presence of drag leads to a drift of the disturbances by the freestream flow along the jet up to their overturning. Figures 3 ($\mu = 10$ P) and 4 ($\mu = 100$ P) show the form of a segment of the jet axis corresponding to one wavelength of disturbance at different instants which are marked by a number by each of the curves. The data presented in figure 3 show that the velocity of the disturbance drift along the jet amounts to approximately 1.5 % of the velocity of motion U_0 of the unperturbed jet. As a matter of fact, in this case the disturbances also represent 'nearly' standing waves in spite of the presence of the drag. Very rapidly ($t = 7$) the jet axis takes the shape of a step, which results in an 'overturn'. During this time the disturbance drifts with the air flow for approximately 0.47 cm and the jet propagates for 33 cm.

The increase in the liquid viscosity with the other parameters left unchanged leads to an increase in the distance for which the disturbance wave propagates along the jet before overturning (see figure 4). The disturbance shape for most of the time before overturning is weakly dependent upon the drag and is mainly determined by the 'lift' component of the aerodynamic force. This is readily understandable, since the drag is quadratic in the disturbance amplitude and, consequently, is significant only for sufficiently large disturbances.

8. Conclusion

An important result of previous calculations is the elucidation of the fact that the bending deformation of a jet of a sufficiently viscous liquid is accompanied by its homogeneous thinning without the localization of deformation (necking) at any point of the jet. Such a synchronous thinning of the jet up to very large amplitudes of the bending wave means that the described quasi-one-dimensional model does not contain the mechanism of the jet disintegration and allows to describe only the predisintegration stage of deformation (see figure 5, corresponding to the moment $t = 7$ of figure 3 and showing the form of the jet segment within one disturbance wavelength). At the same time the available experimental data on high-velocity jets of sufficiently viscous liquids (Haenlein 1932; Grant & Middleman 1966) enable us to conclude that at disturbance amplitudes of the order of several jet radii (2–4) the curved jet breaks up. The flattening of the jet section by a pressure differential existing on it seems to play an important part in this case. Therefore for the description of breakup, the one-dimensional model must be supplemented with equations of the evolution of the jet cross-section. Such estimates based on the energy balance show that after increase in the amplitude of bending disturbances up to the value (2–4) the cross-section of the jet is practically instantaneously flattened by the pressure differential on it. If it is assumed in accordance with the experimental data and the estimates that the breakup occurs instantaneously when the ratio of the bending disturbance amplitude to the jet radius is of the order of several units (2–4), then it is possible to estimate the length of the unbroken part of the jet from the time of growth of disturbances up to the indicated value (and the flattening of the cross-section is insignificant). In this case, as the comparison with the data of numerical calculation shows, the linear theory described in §6 gives an adequate accuracy and it is possible to use (6.21) and (6.23).

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